Homework 2

HW Notes:

- Box your final answer. You will be graded on both the final answer and the steps leading to it. Correct intermediate steps will help earn partial credit.
- For full credit, ~~cross out~~ any incorrect intermediate steps.
- If you need to make any additional assumptions, state them clearly.
- Legible writing will help when it comes to partial credit.
- Simplify your result when possible.

Problems:

1. [3!] Assume a system has the input-output relationship $y(t) = f(t)x(t)$, where $x(t)$ is the input and $y(t)$ is the output. $f(t)$ is not constant, i.e., there exists t_0, t_1 that $f(t_0) \neq f(t_1)$. Show that this system is time-variant. That is, a system with time-variant gain cannot be time-invariant. (*Hint: find a counterexample.*)

2. [9!]Here are input-output relationships for a few systems, all of which are linear. Some of them are time-invariant, some are not. Determine which are which. Find the impulse response of the time-invariant systems.

   (a) $y(t) = \int_{-\infty}^{t} \left[\int_{-\infty}^{s} x(\tau - 5)d\tau\right] ds,$

   (b) $y(t) = \int_{-1}^{3} e^{-(t-\tau)^2} x(\tau)d\tau,$

   (c) $y(t) = \int_{-3}^{3} \tau^2 x(t-\tau)d\tau + \int_{-\infty}^{t+1} (t-\tau+3)^{-2} x(\tau)d\tau.$

   Hint: Note that if you can transform the above relationships into the exact form of convolution $y(t) = g(t) * x(t)$, then the system is immediately time-invariant with $g(t)$ being the impulse response $h(t)$. That is because different from algebraic operators like multiplication, the convolution operator implies time-invariance itself.

3. [9!]One of following two statements is correct, and the other is incorrect. The symbol $\boxed{*}$ denotes *convolution*.

   - If $y(t) = h(t) * x(t)$ then $y(t-3) = h(t-3) * x(t-3)$;
   - Or if $y(t) = h(t) * x(t)$ then $y(t-3) = h(t) * x(t-3)$.

   (a) Give a simple proof of the correct statement.

   (b) Give a simple counterexample for the incorrect statement.

   (c) Repeat (a) and (b) for the following two statements. The symbol $\boxed{\cdot}$ denotes *multiplication*.

   - If $y(t) = h(t) \cdot x(t)$ then $y(t-3) = h(t-3) \cdot x(t-3)$;
   - Or if $y(t) = h(t) \cdot x(t)$ then $y(t-3) = h(t) \cdot x(t-3)$.

   Be careful with the notation $h(t) * x(t)$. More precise notation is $(h * x)(t)$, which makes it clear that convolution is an operation on two signals, not a point-wise operation like multiplication.

4. [8!] Often we will be convolving two signals that are zero everywhere except over some finite range (called **finite support** signals). Suppose $x_1(t)$ is non-zero over the range $a \leq t \leq b$ and that $x_2(t)$ is non-zero over the range $c \leq t \leq d$. Suppose $y(t) = x_1(t) * x_2(t)$.

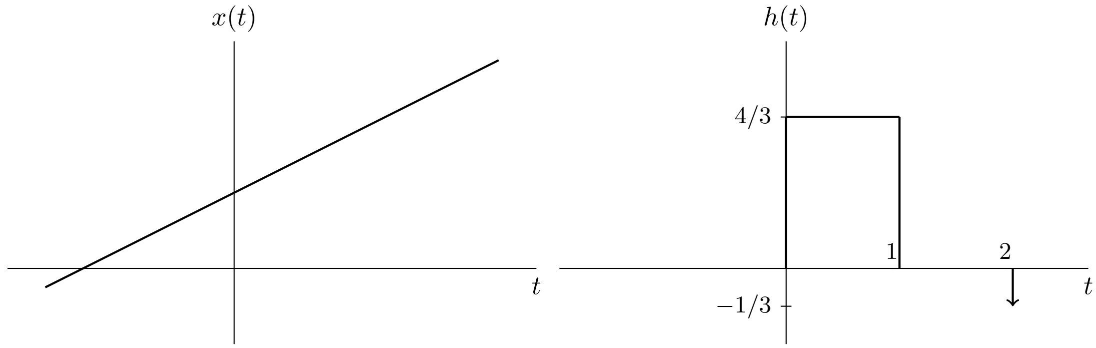
(a) Find the range of values of t for which $y(t)$ is possibly non-zero.

(b) Compute $rect((t-2)/2) * rect((t+3)/4)$ (express answer with braces and carefully sketch). Check your result with part (a).

5. [6!] For each of the following pairs of waveforms, use convolution integral to find the response $y(t)$ of the LTI system with impulse response $h(t)$ and input $x(t)$. Sketch your results.

   (a) $x(t) = e^{-\alpha t}u(t), h(t) = e^{-\beta t}u(t)$ (Do this both when $\alpha = \beta$ and $\alpha \neq \beta$)

   (b) $x(t)$ and $h(t)$ as in the figure below, the slope for the straight line is a and the line intersects y-axis at $(0, b)$:



6. [6!]

   (a) Consider a linear system with input $x(t)$ and output $y(t)$ given by

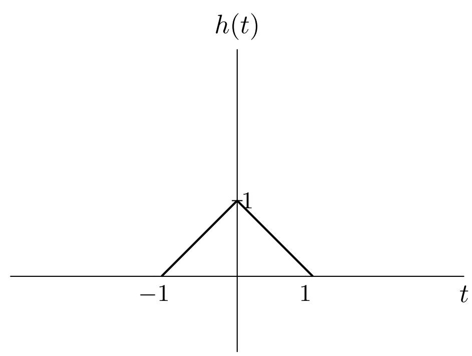
$$y(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT).$$

   Is this system time-invariant?

   (b) Consider another LTI system. Let its impulse response $h(t)$ be the triangular pulse shown below, and $x(t)$ be the **impulse train**

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT),$$

   SKETCH $y(t) = x(t) * h(t)$ for $T = 4, 2, 1.5$ and 1.(No formulae are needed though you still want to label your graphs clearly.)



7. [6!] Let $y(t) = (x * h)(t)$. Show the following properties of convolution.

   (a) $\int_{-\infty}^{\infty} y(t) = \left[\int_{-\infty}^{\infty} x(t)dt\right]\left[\int_{-\infty}^{\infty} h(t)dt\right]$,

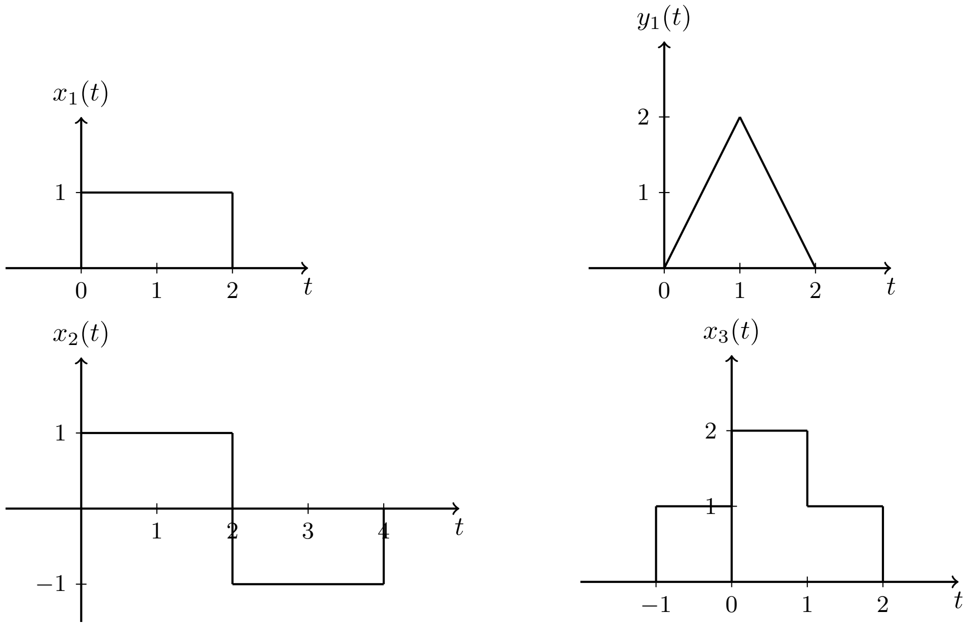
(b) $\frac{d}{dt}y(t) = \left[\frac{d}{dt}x(t)\right] * h(t) = x(t) * \left[\frac{d}{dt}h(t)\right]$,

8. [6!] Compute the following convolution:
   (a) $u(t) * u(t)$,
   (b) $u(t) * t^2u(t)$.

9. [3!] Suppose you have a "black box" LTI system and you discover that for a unit-step input signal the response of the system is the function $(3-t)rect(\frac{t-1}{2})$. Determine the impulse response of the system. (*Hint: See Problem 7.*)

10. [6!] Determine whether the following systems are linear, stable, causal, time-invariant, and memoryless.
   (a) $y(t) = x(\sin(t))$
   (b) $y(t) = \frac{d}{dt}\{e^{-t}x(t)\}$

11. [6!] Consider an LTI system whose response to the signal $x_1(t)$ is the signal $y_1(t)$ which are illustrated below.
   (a) Determine and sketch carefully the response of the system to the input $x_2(t)$ depicted below.
   (b) Determine and sketch carefully the response of the system to the input $x_3(t)$ depicted below.



12. [6!] The triangular pulse is defined as $\text{tri}(t) = (1-|t|)\text{rect}(t/2)$.
   Compute $x(t) = \text{tri}(t/2) * \text{rect}\left(\frac{t-1}{2}\right)$.Express your answer using braces, and carefully sketch.

13. [6!] Find the impulse response of the following LTI systems and further determine whether they are causal, stable and static.
   (a) $y(t) = \int_{-\infty}^{t}(t-\tau)e^{-(t-\tau)}x(\tau)d\tau$
   (b) $y(t) = \int_{t-1}^{t+1}e^{-2(t-\tau)}x(\tau)d\tau$

14. [3!] Consider an LTI system S and a signal $x(t) = 2e^{-3t}u(t-1)$. If

$$x(t) \to y(t)$$

and

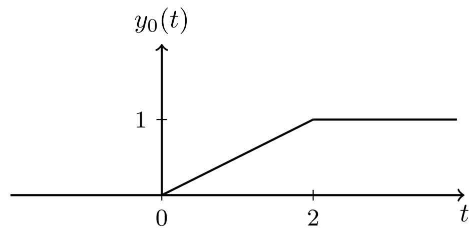
$$\frac{dx(t)}{dt} \to -3y(t) + e^{-2t}u(t)$$

determine the impulse response $h(t)$ of S.

15. [12!] We are given a certain LTI system with impulse response $h_0(t)$. We are told that when the input is $x_0(t)$ the output is $y_0(t)$, which is sketched below. We are then given the following set of inputs to LTI systems with the indicated impulse responses.

	Input $x(t)$	Impulse response $h(t)$
(a)	$x(t) = 2x_0(t)$	$h(t) = h_0(t)$
(b)	$x(t) = x_0(t) - x_0(t-2)$	$h(t) = h_0(t+1)$
(c)	$x(t) = x_0(-t)$	$h(t) = h_0(t)$
(d)	$x(t) = x_0(-t)$	$h(t) = h_0(-t)$
(e)	$x(t) = x_0'(t)$	$h(t) = h_0(t)$
(f)	$x(t) = x_0'(t)$	$h(t) = h_0'(t)$

In each of these cases, determine whether or not we have enough information to determine the output $y(t)$ when the input is $x(t)$ and the system has impulse response $h(t)$. If so, provide an accurate sketch of it with numerical values clearly indicated on the graph.



16. [5!] Find the expression of response of the CT system described by the linear constant-coefficient differential equation.

$$\frac{d}{dt}y(t) + 10y(t) = 2x(t), \qquad y(0) = 1, \qquad x(t) = u(t)$$